Solution of Second Order Nonlinear Singular Value problems

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Abstract

In this paper an endeavor has been put forward to finding a solution of singular nonlinear boundary value problems related to differential equations of second order by Taylor's series method through basic recursive relations of its coefficients exhibiting a sequential spectrums.

Keywords: Singular boundary value problems, He's variational iteration method, Homotopy perturbation method, analytic solution

1. Introduction

The singular nonlinear class of boundary value /initial value problem come across when several problems of many disciplines like mathematics, physics, engineering and interdisciplinary biological and social sciences are taken together and modelled for systemic study and analysis that to accustoming about scientific implications and specific behaviour eventually. Such phenomenon expressed mathematically challenge and motivate the world of science to explore a well-defined solution all over. However it is desirable and must to acknowledge that modelling particularly physical events and observations in Astrophysics, Electro hydrodynamics and Thermal explosions produce to have a general form of singular two point value problems of differential equation of second order[2,3,4,8,10,11,12,16,19] as follows

$$\frac{1}{p(x)}y''(x)+\frac{1}{q(x)}y'(x)+\frac{1}{r(x)}\;y(x)=f(x) \qquad\qquad 0<x\leq 1 \qquad (1.1)$$

Subject to the boundary conditions

$$y(0)=A_1, \qquad y(1)=B_1 \qquad or \qquad y'(0)=A_2, \qquad y(1)=B_2$$

Where p, q, r, f are continuous functions of x defined over [0, 1] and A_1, A_2, B_1, B_2 are real numbers. Determining a solution of such type of problems is very important as they have wide applications in scientific and engineering applications such as boundary value theory, flow networks of biology, control and optimization theory etc. Many scientists have discussed various methods for obtaining their numerical solutions that include B-splines [6, 7 , 26]. Divided difference method, perturbation methods [21].Adomian decomposition method [1].Modified decomposition method [29]. Projection method [27]. One of the important methods that has received attention in literature recently is He's polynomial method [1, 6]. Ghorbani is one who for the first time introduced He's polynomials [21]. The He's polynomials are calculated using the He's Homotopy perturbation method [23, 25].

2. Method of solution

One of the common techniques for solving singular boundary value problems is that the original differential equation is represented differently at singular point and at other points, i.e., non-singular points in the given interval in its original form. The basic purpose to write the differential equation in modified way is to handle singularity involved that behaves viciously and unexpectedly. However, after proper and suitable treatment numerical method such as Newton's divided difference or any other method can be applied to obtain a desired numerical solution. In this paper, we discuss a simple method for obtaining numerical solutions of the singular boundary value problems. The results may be compared with that to any of the henceforth described methods for example He's polynomial method [1]. The proposed method provides solutions more accurate like any other good method available in literature. Consider the problem (1.1), which can be modified and rewritten as

$$q(x)r(x)y''(x)+p(x)r(x)y'(x)+p(x)q(x)y(x)=p(x)q(x)r(x)f(x) \quad 0\le x\le 1 \qquad (2.1)$$

Subject to the boundary conditions

$$y(0)=A_{1,}\ y(1)=B_1 \quad or \quad y'(0)=A_2\ , y(1)=B_2$$

We apply the Leibnitz rule for differentiating the product terms multiple times (say n) occurring in equation (2.1). The value of ' n' is determined by the order of accuracy of the desired results.

$$\sum_{k=0}^{n} C_k^n\,(q(x)r(x))^{(n-k)}\,y^{(k+2)}(x)+\sum_{k=0}^{n} C_k^n\,(p(x)r(x))^{(n-k)}y^{(k+1)}(x)$$

$$+\sum_{k=0}^{n} C_k^n(p(x)q(x))^{(n-k)}y^{k}(x)=(p(x)q(x)r(x)f(x)\ )^{(n)} \qquad (2.2)$$

Eqn. (2.2) can also be written as

$$\sum_{k=0}^{n-1} C_k^{n-1}(q(x)r(x))^{(n-1-k)}y^{(k+2)}(x)-\sum_{k=2}^{n} C_k^n\,(p(x)r(x))^{n-k}y^{(k+1)}(x)+$$
$$\sum_{k=0}^{n} C_k^n\,(p(x)q(x))^{(n-k)}y^{(k)}(x)=(p(x)q(x)r(x)f(x))^{n} \qquad (2.3)$$

We can write the above eqn. (2.3) as follows

$$y^{(n+2)}(x)=\ -\sum_{k=0}^{n-1} C_k^{n-1}(q(x)r(x))^{(n-1-k)}y^{(k+2)}(x)-\sum_{k=0}^{n} C_k^n\,(p(x)r(x))^{(n-k)}y^{(k+1)}(x)-$$
$$\sum_{k=0}^{n} C_k^n(p(x)q(x))^{(n-k)}y^{(k)}(x)=(p(x)q(x)r(x)f(x))^{(n)}$$

The above equation is defined for all values of x in [0, 1]. We take any point in this interval. Taking x=0, and n=0, 1, 2 … we obtain $y^{(2)}(0), y^{(3)}(0), \dots, y^{(n)}(0), \dots$ .some of these values may be unknown. Indeed, some of them are unknown and some others may depend on these unknown values. We can expand y(x) as a Taylor series in the neighbourhood of a point x=0 in the interval [0, 1] i.e.

$$y(x)=y(x_0)+y'(x_0)+\frac{1}{2!}y''(x_0)(x-x_0)^2+\frac{1}{3!}y'''(x_0)\,(x-x_0)^3+\frac{1}{4!}y^{iv}(x_0)(x-x_0)^4+\cdots\dots\dots\dots\dots$$

$$=\ y(x_0)+\ \sum_{n=1}^{\infty}\frac{1}{n!}y^{(n)}(x_0)(x-x_0)^n \qquad (2.4)$$

Solution to such class of problems (1.1) exists and is unique [14,24,28].Therefore the well desired solution to the given boundary value problem (1.1) of second order non-singular differential equation can be felicitated by

$$y(x) = y(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} y^{(n)}(x_0)(x - x_0)^n + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$

Using the given boundary conditions in above equation (2.4).Now every unknown value of the solution numerically or exact solution can be found depending upon the given problem. Once all values of the derivatives through recurrence relation are made known i.e.

$y^{(2)}(0), y^{(3)}(0), \ldots, y^{(n)}(0)$ are known after simplification, we can compute the numerical value of y at any point in the interval [0, 1] .

3. Illustrative Examples

In this section with regard to finding the solution of nonlinear singular value problems and to test the efficiency and efficacy of our proposed method after a sequence of Taylor series coefficient is procured successfully. As a derived outcome it is pleasing to have an exact solution in the end.

3.1 Example

Consider the nonlinear homogeneous boundary value problem[9,15]

$$y''(x) + \frac{5}{x} y'(x) + 8(e^{y(x)} + 2e^{y(x)/2}) = 0 \qquad 0 < x \leq 1 \qquad (3.1.1)$$

Subject to boundary condition

$y'(0) = 0$ And $y(1) = -2ln2$

Solution:

Now we modify equation (3.1.1) to find its solution in the following manner

$$x\, y''(x) + 5y'(x) + 8x(e^{y(x)} + 2e^{y(x)/2}) = 0 \qquad (3.1.2)$$

Now taking the limit as ($x \to 0$)

We get, $y'(0) = 0$ (3.1.3)

Differentiating eqn. (3.1.2) once and putting x=0 we have

$$y^{(2)}(0) + \frac{4}{3}(e^{y(0)} + 2e^{y(0)/2}) = 0 \qquad (3.1.3)$$

Differentiating (3.1.2) twice and then taking the limit as($x \to 0$)

We get, $y^{(3)}(0) = 0$ (3.1.4)

Differentiating (3.1.2) thrice and taking limit as ($x \to 0$) we get

$$y^{(4)}(0) + 3\left(e^{y(0)} + e^{ya(1)/2}\right) y^{(2)}(0) = 0 \qquad (3.1.5)$$

Differentiating (3.1.2) four times and taking limit as (x→0)

We get, $y^5(0) = 0$ (3.1.6)

Differentiating (3.1.2) five times and taking limit as (x→0), we have

$$y^{(6)}(0) + 12(y^{(2)}(0))^2\left(e^{y(0)} + \frac{1}{2}e^{y(0)/2}\right) + 4(e^{y(0)} + e^{y(0)/2})y^{(4)}(0) = 0 \quad (3.1.7)$$

Differentiating (3.1.2) six times and taking limit as (x→0)

We get, $y^{(7)}(0) = 0$ (3.1.8)

That we have upon differentiating seven times and taking limit as (x→0),

$$y^{(8)}(0) + 70(e^{y(0)} + \frac{1}{4}e^{y(0)/2})(y^{(2)}(0))^2 + 70\left(e^{y(0)} + \frac{1}{2}e^{y(0)/2}\right)y^{(2)}(0)y^{(4)}(0)$$

$$+ \frac{14}{3}\left(e^{y(0)} + e^{y(0)/2}\right)y^{(4)}(0) = 0 \quad (3.1.9)$$

Now it may be noted that all the odd derivatives appearing in the Taylors series vanishes at(x=0)

However, these recurrence relations simplifies to give

$$y^{(2)}(0) = -4/3\left(e^{y(0)} + 2e^{y(0)/2}\right) \quad (3.1.10)$$

$$y^{(4)}(0) = 4(e^{y(0)} + e^{y(0)/2})(e^{y(0)} + 2e^{y(0)/2}) \quad (3.1.11)$$

$$y^{(6)}(0) = -\frac{16}{3}(e^{y(0)} + 2e^{y(0)/2})(7e^{y(0)} + 16e^{y(0)/2} + 7)e^{y(0)} \quad (3.1.12)$$

$$y^{(8)}(0) = 2\times 8\times 7[(e^{y(0)} + 2e^{y(0)/2})/3][(443e^{2y(0)} + 382e^{y(0)} + 623e^{3y(0)/2} + 172e^{y(0)/2})]/3^2]e^{y(0)} \quad (3.1.13)$$

Therefore Taylor series solution of above differential equation is given by

$$y(x) = y(0) - 2[(e^{y(0)} + 2e^{y(00/2})/3]x^2/_1 + 2[(e^{y(0)} + 2e^{y(00/2})/3]((x^2)^2/2)(e^{y(0)} + e^{y(0)/2})/2$$

$$-2(e^{y(0)} + 2e^{y(00/2})/3]((x^2)^3/3)\,[(7e^{y(0)} + 16e^{y(0)/2} + 7)/30]e^{y(0)}$$

$+2(e^{y(0)}+2e^{y(00}/2)/3]((x^2)^4/4)\ [(443e^{2y(0)}+172e^{y(0)/2}+382e^{y(0)})/1620]e^{y(0)}$

$+$……………………………………………………………………….. (3.1.14)

Now strength of nonlinearity is mitigated and marginalised in the existing recurrence relations appearing for derivatives of solution function at origin by taking '$y(0)=0$',even so the solution shoots to satisfy the other extreme boundary condition '$y(1)=-2ln2$' equivalently giving $e^{y(0)}=1$.

Thereby giving as a matter of fact,$y(x)=-2\ln(1+x^2)$ as the desired solution to boundary value problem.

3.2Example

Consider the nonlinear differential equation[9,15]

$$y''(x)+\frac{8}{x}y'(x)+18y(x)+4y(x)lny(x)=0 \quad , \quad 0< x\leq 1 \qquad (3.2.1)$$

Subject to boundary conditions $y'(0)=0$ And $y(1)=\exp(1)$

Solution: Modify (3.2.1) to adjust singularity at (x=0) as

$$xy''(x)+8\,y'(x)+18xy(x)+4xy(x)ln\,y(x)=0 \quad , \quad 0\leq x\leq 1 \qquad (3.2.2)$$

Taking limit as (x→0)

We get,$y'(0)=0$ (3.2.3)

Now differentiating (3.2.2) with respect to 'x' and taking (x→0), we have

$$9y''(0)+18y(0)=-4y(0)ln\,y(0) \qquad (3.2.4)$$

Differentiating (3.2.2) with respect to respect to 'x' twice and taking (x→0)

We get, $y^{(3)}(0)=0$ (3.2.5)

Differentiating (3.2.2) with respect to respect to 'x' thrice and taking (x→0), we get

$$11y^{(4)}(0)+66y''(0)=-12y''(0)ln\,y(0) \qquad (3.2.6)$$

Differentiating (3.2.2) with respect to respect to 'x' four times and taking (x→0)

We get, $y^{(5)}(0) = 0$ (3.2.7)

Differentiating (3.2.2) with respect to respect to 'x' five times and taking (x→0), we get

$$13y^{(6)}(0) + 110y^{(4)}(0) + 60\,(y^{(2)}(0))^2/y(0) = -20y^4(0)\ln y(0) \quad (3.2.8)$$

Differentiating (3.2.2) with respect to respect to 'x' six times and taking (x→0)

We get, $y^{(7)}(0) = 0$ (3.2.9)

Differentiating (3.2.2) with respect to respect to 'x' seven times and taking (x→0), we get

$$15y^{(8)}(0) + 154y^{(6)}(0) + 420\,y^{(2)}(0)y^{(4)}(0)/y(0) - 420\,((y^{(2)}(0))^3/y(0)^2 = -28\,y^{(4)}(0)\ln y(0) \quad (3.2.10)$$

Now it is fair to observe that strength of nonlinearity in the henceforth deduced recurrence relations involving the various derivatives is minimised and simplified by choosing $y(0) = 1$, so that $\ln y(0) = 0$. Without loss of generality and in view of an anticipated solution whatsoever that may, it is still possible to fulfils all other required criterion by the solution like that further shoots to satisfy the specified and given next extreme boundary condition.

Under imposed circumstances, relations (3.2.4), (3.2.6), (3.2.8) and (3.2.10) simplifies to produce $y''(0) = -2$, $y^{(4)}(0) = 12$, $y^{(6)}(0) = -120$, $y^{(8)}(0) = 1680$-------------------------------- (3.2.11) continuing similarly we may have eventually that

$y^{(2n)}(0) = (-1)^n \prod_{j=n+1}^{2n}(j)$ for all n≥ 5 and all the odd derivatives turn out to be zero. Which in turn produces $y(x) = e^{-x^2}$ as the required solution to the boundary value problem (3.2.1).

3.3Example

Consider a singular second order differential equation arising in astronomy modelling the equilibrium of isothermal gas sphere can be described by [25]

$$y''(x) + y'(x)/x - (y^3(x) - 3y^5(x)) = 0 \quad (3.3.1)$$

Subject to conditions $y'(0) = 0$, $y(1) = \frac{1}{\sqrt{2}}$

Solution: Now in order to deal with singularity at 'x=0' we modify (3.3.1) as follows

$$xy''(x) + 2y'(x) + xy^5(x) = 0 \quad (3.3.2)$$

It may be noted that (3.3.2) is well defined for all values in [0, 1]. Using $x \to 0$ in (3.3.2), we have $y'(0) = 0$, which is the first boundary condition. Now differentiating (3.3.2) with respect to 'x' and taking the limit as (x→0) we get

$$2y^{(2)}(0) = (y(0))^3[(1 - 3\big(y(0)\big)^2]/2 \qquad (3.3.3)$$

Differentiating (3.3.2) with respect to 'x' twice and taking the limit as (x→0)

We get $y^{(3)}(0) = 0$ (3.3.4)

Again differentiating (3.3.2) with respect to 'x' thrice and taking the limit as (x→0) we get

$$y^{(4)}(0) = 3^2(\frac{1-5y^2(0))}{4}y^2(0) \quad y^2(0) \qquad (3.3.5)$$

Now differentiating (3.3.2) with respect to 'x' four times and taking limit as (x→0) we get

$$y^{(5)}(0) = 0 \qquad (3.3.6)$$

Differentiating again (3.3.2) with respect to 'x' five times and taking limit as (x→0) we get

$$2y^{(6)}(0) = 5[6y(0)(y^{(2)}(0))^2 + \big(y(0)\big)^2 y^{(4)}(0) - 60y^3(0)(y^{(2)}(0))^2 - 5y^4(0)y^{(4)}(0)] \qquad (3.3.7)$$

Again differentiating (3.3.2) with respect to 'x' six times and taking limit as (x→0) we get

$$y^{(7)}(0) = 0 \qquad (3.3.8)$$

Now differentiating (3.3.2) with respect to 'x' seven times and taking limit as (x→0) we get

$$8y^8(0) = 7[90(y^2(0))^3 + 90y^{(2)}(0)y^{(4)}(0) + 3y^2(0)y^{(6)}(0) - 2700y^2(0)(y^2(0))^3 - 900y^3(0)y^2(0)y^4(0) - 15y^4(0)y^{(6)}(0)]$$

Now, since putting $y(0) = 1$ in these recurrence relations does not change the matching value of the solution at right boundary. Therefore doing so the relations are simplified to produce

$$y^{(2)}(0) = -1, y^{(3)}(0) = 0,\ y^{(4)}(0) = 3^2,\ y^{(5)}(0) = 0\,, y^{(6)}(0) = -3^2.5^2, y^{(7)}(0) = 0 \qquad y^{(8)}(0) = 3^2.5^2.7^2$$

$y^{2n}(0) = (-1)^{n\prod_{j=3}^{2n-1} j^2}$ For every n≥5

Thus Taylor series method yields exact solution to the given boundary value problem (3.4.1), as

y(x) =1+ $(-\frac{1}{2})(x^2/3)$+$(-\frac{1}{2})(-\frac{3}{2})(x^2/3)^2+(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(x^2/3)^3+(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})(x^2/3)^4$+.......

That is,$y(x) = 1/\sqrt{1+(x^2/3)}$ is the required solution of the boundary value problem (3.3.1).

3.4Example

Consider a singular second order differential equation arising in astronomy modelling the equilibrium of isothermal gas sphere can be described by[25]

$$y''(x)+\frac{2}{x}y'(x)+y^5(x)=0 \qquad (3.4.1)$$

Subject to conditions $y'(0)=0$, $y(1)=\frac{1}{2}\sqrt{3}$

Solution:

Now in order to deal with singularity at 'x=0' we modify (3.3.1) as follows

$$xy''(x)+2y'(x)+xy^5(x)=0 \qquad (3.4.2)$$

Now taking the limit as (x→0) implies that

$y'(0)=0$ Differentiating (3.4.2) with respect to 'x' and taking limit as (x→0) we get

$$y^{(2)}(0)=-y^5(0)/3 \qquad (3.4.3)$$

Differentiating (3.4.2) with respect to 'x' twice and taking limit as (x→0) we get

$$y^{(3)}(0)=0 \qquad (3.4.4)$$

Differentiating (3.4.2) with respect to 'x' thrice and taking limit as (x→0) we get

$$y^{(4)}(0)=-3y^4(0)y^{(2)}(0) \qquad (3.4.5)$$

Differentiating (3.4.2) with respect to 'x' four times and taking limit as (x→0) we get

$$y^{(5)}(0)=0 \qquad (3.4.6)$$

Once again differentiating (3.4.2) with respect to 'x' five times and taking limit as (x→0) we get

$$7y^{(6)}(0) = 5^2(12(y^{(2)}(0))^2 + y(0)y^{(4)}(0)y^3(0)) \qquad (3.4.7)$$

Differentiating again (3.4.2) with respect to ‘x’ six times and taking limit as (x→0) we get

$$y^{(7)}(0) = 0 \qquad (3.4.8)$$

Again differentiating (3.4.2) with respect to ‘x’ seven times and taking limit as (x→0) we get

$$9y^{(8)}(0) + 7(900y^2(0)(y^{(2)}(0))^3 + 300y^3(0)y^{(2)}(0)y^{(4)}(0) + y^4(0)y^{(6)}(0)) = 0 \qquad (3.4.9)$$

Now the recurrence relations interconnecting the derivatives are semi linearized and simplified by taking $y(0) = 1$ without affecting the value of solution at the right boundary ($y(1) = \sqrt{3}/2$) so as to produce after simultaneous simplifications of recurrence relations

$$y^{(2)}(0) = -{}^{1}/_{3}\, , y^{(3)}(0) = 0, y^{(4)}(0) = 1, y^{(5)}(0) = 0, y^{(6)}(0) = 25/3, y^{(7)}(0) = 0, y^{(8)}(0) = 875/27 \quad (3.4.10)$$

Therefore the solution of the boundary value problem by Taylor series method is given by

$$y(x) = 1 - \frac{1}{6}x^2 + \frac{1}{24}x^4 - \frac{5}{756}x^6 + \frac{5}{7776}x^8 + \cdots = 1/\sqrt{(1 + (x^2/3)} \qquad (3.4.11)$$

3.5Example

consider the boundary value problem

$$y''(x) + \frac{2}{x} \qquad y'(x) + y^2(x) \quad = \frac{2}{x} + x^2, \quad 0 < x \le 1 \qquad (3.5.1)$$

Subject to conditions $y'(0) = 1$ and $y(1) = 1$

Solution: In order to solve (3.5.1) we modify the given relation equivalently as

$$. \, y''(x) + 2y'(x) + xy^2(x) \qquad = 2 + x^3 \qquad (3.5.2)$$

Now (3.5.2) is well defined for all the values in the interval [0 1]

Obviously (3.5.2) after taking limit as(x→0) implies and satisfies the first boundary condition

Further, differentiating (3.5.2) with respect to ‘x’ and taking limit as(x→0) we get

$$y^{(2)}(0) = -y^2(0)/3 \qquad (3.5.3)$$

Again, differentiating (3.5.2) with respect to 'x' twice and taking limit as($x\rightarrow 0$) we get

$$y^{(3)}(0) = -y(0) \qquad (3.5.4)$$

Again, differentiating (3.5.2) with respect to 'x' thrice and taking limit as($x\rightarrow 0$) we get

$$y^{(4)}(0) = {}^{2}/_{5}\, y^{3}(0) \qquad (3.5.5)$$

Again, differentiating (3.5.2) with respect to 'x' four times and taking limit as ($x\rightarrow 0$) we get

$$3y^{(5)}(0) = -4(1 + y(0))y(0) \qquad (3.5.6)$$

Again, differentiating (3.5.2) with respect to 'x' five times and taking limit as ($x\rightarrow 0$) we get

$$7y^{(6)}(0) = -\frac{40}{3}\, y^{3}(0) + 32\, y(0) - 4\, y^{3}(0) \qquad (3.5.7)$$

Again, differentiating (3.5.2) with respect to 'x' six times and taking limit as ($x\rightarrow 0$) we get

$$8y^{(7)}(0) + 112\, y^{(2)}y^{3}(0) + 34\, y^{1}(0)y^{(0)}(0) + 46\ \ y^{(2)}(0) + 8(y^{3}(0))^{2} + 14\, y^{(1)}(0)y^{(5)} + 2\, y(0)y^{5}(0) + y(0)y^{(6)}(0) = 0$$

(3.5.8)

Now putting the value of derivatives of y at zero after being simplified from (3.5.3) , (3.5.4) , (3.5.5) , (3.5.6) , (3.5.7) ,and (3.5.8) in the Taylor series solution (2.4) and imposing the other extreme boundary condition y(1)=1, we get y(0)=0, thereby giving y(x)=x as the exact solution to the problem (3.5.1).

5. Conclusion

It is noteworthy to observe that the Taylor series solution method is equally efficient and can be trusted as the other methods of solutions do, when eventually somehow a sequence of Taylor series coefficient are extracted properly. Over all as a matter of fact the proposed method may fair well if applied on some of the prime problems available throughout the literature successfully to have had generated even exact solution.

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